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ON THE EIGENVECTORS OF THE MATRIX THAT PERFORMS THE DISCRETE FINITE FOURIER TRANSFORM

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ON THE EIGENVECTORS OF THE MATRIX THAT PERFORMS THE DISCRETE FINITE FOURIER TRANSFORM

BY

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ABSTRACT

The discrete finite Fourier transform can be regarded as a matrix operation, since each element of one member of the pair is a linear combination of all the elements of the other member. The N-by-N cyclotomic matrix $(W)_{jk} = N^{-\frac{1}{2}} \exp[-2\pi ijk/N]$ which performs the transform is unitary and has eigenvalues $\lambda = \pm 1$ and $\pm i$. Clearly the eigenvectors of W are those functions which are their own finite Fourier transform multiplied by ± 1 or $\pm i$. One class of such functions are aliased Hermite functions, which are related to the theta functions.

We discuss analytically some curious properties of these functions, which were suggested by numerical calculations of the eigenvectors.

We demonstrate a remarkably simple relation between a periodic function of a discrete variable and its discrete finite Fourier transform, namely that the absolute values of their expansion coefficients in these eigenvectors are the same. We suggest a canonical form for such functions (with respect to the finite Fourier transform) in which the transform can be done by inspection.

TABLE OF CONTENTS

1. Introduction

Popularization of the fast Fourier transform algorithm by Cooley and others (for example, Cooley et al., 1967) has caused renewed interest in the theory of the discrete finite Fourier transform, which was neglected during the years when statisticians preferred to calculate the spectra of digital time series via mean lagged products. In this paper we discuss some properties of the eigenvectors of the matrix that performs the discrete finite Fourier transform.

For continuous time we have the Fourier transform pair:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi i f t}\, dt \qquad (1.1)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+2\pi i f t}\, df \qquad (1.2)$$

provided these integrals exist. The corresponding discrete finite Fourier transform pair is:

$$X_k = N^{-\frac{1}{2}} \sum_{j=0}^{N-1} x_j\, e^{-2\pi i j k/N} \qquad k = 0, N-1 \quad (1.3)$$

$$x_j = N^{-\frac{1}{2}} \sum_{k=0}^{N-1} X_k\, e^{+2\pi i j k/N} \qquad j = 0, N-1 \quad (1.4)$$

for which the completeness-closure relation is:

$$N^{-1} \sum_{j=0}^{N-1} \exp[2\pi ij(k - k')/N] = \delta_{k'}^{k}, \qquad (1.5)$$

where

$$\delta_{k'}^{k} = \begin{cases} 1 \text{ when } k \equiv k' \pmod{N} \\ 0 \text{ otherwise} \end{cases} \qquad (1.6)$$

The relation between the discrete finite Fourier transform and the continuous-time Fourier transform is (see, for example, Cooley et al., 1967) that if $x(t)$ and $X(f)$ are a continuous-time Fourier transform pair, then $x_p(\Delta t)$ and $X_p(k\Delta f)$ are a discrete finite Fourier transform pair, where

$$x_p(j\Delta t) = T\, N^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} x(j\Delta t + mN\Delta t) \qquad (1.7)$$

$$X_p(k\Delta f) = \sum_{m=-\infty}^{\infty} X(k\Delta f + mN\Delta f) \qquad (1.8)$$

where N is the number of sample points in x and X, and $T = N\Delta t$. The form of (1.7) and (1.8) is obviously due to the aliasing in both time and frequency which occurs when $x(t)$ and $X(f)$ are sampled.

We define a matrix W:

$$(W)_{jk} = N^{-\frac{1}{2}}\, e^{-2\pi ijk/N} \qquad (1.9)$$

in order to write (1.3) and (1.4) more compactly in matrix notation:

$$\underset{\sim}{X} = W \underset{\sim}{x} \tag{1.10a}$$

$$\underset{\sim}{x} = W^{\dagger} \underset{\sim}{X} \tag{1.10b}$$

where $\underset{\sim}{x}$ and $\underset{\sim}{X}$ are vectors whose elements are the sample values of the data and the transform respectively, and the dagger denotes Hermitian conjugation. W is clearly unitary, and it satisfies the equation

$$W^4 = I \tag{1.11}$$

so that its eigenvalues are $\lambda = \pm 1$ and $\pm i$. The problem of determining the multiplicity of these eigenvalues is equivalent to the problem of evaluating the trace of W, i.e., the Gaussian sum

$$S(1,N) = N^{-\frac{1}{2}} \sum_{k=1}^{N} \exp(2\pi i k^2/N) = N^{\frac{1}{2}} \operatorname{tr}(W)$$

the value of which is well known (Erdélyi et al., 1955, section 17.6).

Carlitz (1959) showed that a multiplicity rule for the eigenvalues can be derived from the fact that the characteristic polynomial of W is:

$$f(\lambda) = (\lambda - 1)^2(\lambda + i)(\lambda + 1)(\lambda^4 - 1)^{\frac{1}{4}N-1} \qquad \text{when } N \equiv 0 \pmod 4$$

$$f(\lambda) = (\lambda - 1)(\lambda^4 - 1)^{\frac{1}{4}(N-1)} \qquad N \equiv 1 \pmod 4$$

$$f(\lambda) = (\lambda^2 - 1)(\lambda^4 - 1)^{\frac{1}{4}(N-2)} \qquad N \equiv 2 \pmod 4$$

$$f(\lambda) = (\lambda + i)(\lambda^2 - 1)(\lambda^4 - 1)^{\frac{1}{4}(N-3)} \qquad N \equiv 3 \pmod 4$$

A simple way to see the multiplicity is to count the occurrences of each of the fourth roots of unity around the unit circle in the following way: For N odd, start with +1 and step around counterclockwise, counting each root in turn, up to a total of N of them - e.g., for N = 5 the eigenvalues are 1, -i, -1, +i, 1. For N even, however, the last occurrence of i or -i is skipped - e.g., for N = 6 the eigenvalues are 1, -i, -1, +i, 1, -1.

2. The Eigenvectors of W

It is clear that the eigenvectors of W are those functions of a discrete variable which are equal to their own discrete finite Fourier transforms multiplied by ± 1 or $\pm i$. We can get a set of these from the relationship (see Magnus et al., 1966):

$$\int_{-\infty}^{\infty} \{e^{-\pi t^2} H_k[(2\pi)^{\frac{1}{2}} t]\} e^{-2\pi i f t}\, dt = i^{-k} \{e^{-\pi f^2} H_k[(2\pi)^{\frac{1}{2}} f]\} \quad (2.1)$$

where $H_k(x)$ is the k'th Hermite polynomial:

$$H_k(x) = (-1)^k \exp(x^2)\, d^k[\exp(-x^2)]/dx^k \quad (2.2)$$

That is,

$$u(t) = e^{-\pi t^2} H_k[(2\pi)^{\frac{1}{2}} t] \quad (2.3)$$

is a solution of the singular integral equation:

$$\int_{-\infty}^{\infty} u(t)\, e^{-2\pi i f t}\, dt = \lambda u(f) \quad (2.4)$$

with $\lambda = \pm 1, \pm i$.

The functions (2.3) do not exhaust the solutions of (2.4), since there are infinitely many solutions for $\lambda = \pm 1$: for example, $u(t) = \operatorname{sech}(\pi t)$ satisfies (2.4) with $\lambda = 1$, and in fact, if $x(t)$ and $X(f)$ are any Fourier transform pair, then

$$u(t) = x(t) \pm \alpha X(t) \tag{2.5}$$

is a solution of equation (2.4), with $\lambda = \alpha = \pm 1$ if $x(t) = x(-t)$ and with $\lambda = -\alpha = \pm i$ if $x(t) = -x(t)$.

is a solution of (2.4) with $\lambda = \pm 1$.

From equations (1.7) and (1.8) we see that eigenvectors of W which correspond to the functions (2.3) are:

$$u_k(j) = \sum_{m=-\infty}^{\infty} e^{-\pi(j + mN)^2/N} H_k[(2\pi/N)^{\frac{1}{2}}(j + mN)] \tag{2.6}$$

where $u_k(j)$ denotes the j'th element of the k'th eigenvector. In deriving (2.6) we made use of the fact that if $x_p(j\Delta t)$ is to be proportional to $X_p(k\Delta f)$, then $\Delta t = \Delta f = N^{-\frac{1}{2}}$. Cooley et al. (1967) state (2.6) omitting the Hermite function. Since $H_0(t) = 1$, (2.6) is a generalization of their result.

These functions of j, k, and N are clearly periodic in j with period N; for even k the functions are symmetric about the middle of a period and for odd k they are antisymmetric. McClellan and Parks (1972) pointed out that this is a general property of the eigenvectors of W.

An easy way to see this general property follows from the fact that $\lambda = \pm 1$, $\pm i$ and

$$(W^2)_{kj} = \delta^k_{N-j}, \qquad k,j = 0,\ 1,\ldots,\ N-1 \tag{2.7}$$

where δ^k_{N-j} is the Kroneker delta defined in equation (1.6).

Thus we have:

$$W^2\ \mathrm{col}(u_0,\ldots,u_N) = \pm\ \mathrm{col}(u_0,\ldots,u_N) \tag{2.8}$$

where the sign is positive for $\lambda = \pm 1$ and negative for $\lambda = \pm i$. From (2.7) it is clear that

$$\begin{aligned} u_j &= u_{N-j} && \text{for } \lambda = \pm 1 \text{ and } j = 1,\ N-1 \\ u_j &= -u_{N-j} && \text{for } \lambda = \pm i \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} u_0 &= 0 && \text{for } \lambda = \pm i \\ u_{N/2} &= 0 && \text{for } \lambda = \pm i \text{ and } N \text{ even} \end{aligned} \tag{2.10}$$

We see from (2.1) that the order index k is related to the eigenvalues: the functions (2.6) are eigenvectors of W with eigenvalues

$$\lambda = \begin{cases} 1 & \text{for } k \equiv 0 \pmod 4 \\ -i & k \equiv 1 \pmod 4 \\ -1 & k \equiv 2 \pmod 4 \\ i & k \equiv 3 \pmod 4 \end{cases} \tag{2.11}$$

In this paper we do not consider other functions which satisfy (2.4), for example

$$u(j) = \sum_{m=-\infty}^{\infty} \operatorname{sech}[\pi N^{-\frac{1}{2}}(j+mN)] \tag{2.12}$$

(2.6) is a sequence of functions which are multiples of the eigenvectors. Since there are infinitely many functions in the sequence, clearly there are many linear dependences. The functions in any infinite sequence associated with a given eigenvalue of W must also contain many linear dependences, since the multiplicity of the eigenvalue is a bound on the number of linearly independent eigenvectors associated with that eigenvalue.

We notice that in the discrete case the property (2.5) also holds true.

There is an interesting relation between the eigenvectors of W and a function which occurs in number theory: the Legendre-Jacobi symbol is defined as:

$$\left(\frac{m}{N}\right)=\begin{cases} 1 & \text{when } m \equiv z^2 \pmod{N} \neq 0 \\ -1 & \text{when } m \equiv z^2 \pmod{N} \neq 0 \\ 0 & \text{when } m \equiv 0 \pmod{N} \end{cases} \tag{2.13}$$

where z is an integer (Erdélyi <u>et al</u>., 1955, section 17.5). It can be shown that

$$\begin{aligned} &\left(\frac{m}{N}\right) \varepsilon\ E_1^{(N)} \qquad \text{for } N \equiv 1 \pmod{4} \\ &\left(\frac{m}{N}\right) \varepsilon\ E_{-i}^{(N)} \qquad \text{for } N \equiv 3 \pmod{4} \end{aligned} \tag{2.14}$$

where $E_\lambda^{(N)}$ is the vector space spanned by the eigenvectors associated with the eigenvalue λ.

3. Some general properties of the eigenvectors of W

We calculated the functions (2.6) to twenty significant figures for N = 2, 3, ..., 30 and k = 0, 1, ..., 60. A number of interesting properties emerged from the calculations, which led to the following general conclusions.

We consider the even vectors

$$\{\delta^k_{j_0} + \delta^k_{N-j_0}\} \qquad j_0 = 0,1,\ldots,\tfrac{1}{2}(N - 1 + N_2) \tag{3.1}$$

and the odd vectors

$$\{\delta^k_{j_0} - \delta^k_{N-j_0}\} \qquad j_0 = 1,\ldots,\tfrac{1}{2}(N - 1 - N_2)$$

where $N_2 \equiv N \pmod 2 \geq 0$. With these vectors we form the following total systems of vectors (which are, however, not minimal systems) in the spaces $E^{(N)}_\lambda$:

$$N^{-\frac{1}{2}} \cos(2\pi j_0 k/N) \pm \tfrac{1}{2}(\delta^k_{j_0} + \delta^k_{N-j_0}) \in E^{(N)}_{\pm 1} \tag{3.2}$$

and

$$N^{-\frac{1}{2}} \sin(2\pi j_0 k/N) \pm \tfrac{1}{2}(-\delta^k_{j_0} + \delta^k_{N-j_0}) \in E^{(N)}_{\pm i} \tag{3.3}$$

These systems contain almost twice as many vectors as the dimensionality of $E^{(N)}_\lambda$; orthogonalizing these systems separately for each λ it is possible to get a full system of orthonormal eigenvectors of W.

Consider the vectors with elements $j_0 = 0$ from $E_{\pm 1}^{(N)}$ and with element $j_0 = 1$ from $E_{\pm i}^{(N)}$. Then for any N,

$$\mathrm{col}(1 \pm N^{\frac{1}{2}}, 1, \ldots, 1) \in E_{\pm 1}^{(N)} \tag{3.4}$$

and

$$\mathrm{col}(0, \alpha_1 \mp N^{\frac{1}{2}}, \alpha_2, \ldots, \alpha_{N-2}, \alpha_{N-1} \pm N^{\frac{1}{2}}) \in E_{\pm i}^{(N)} \tag{3.5}$$

where

$$\alpha_k = 2 \sin(2\pi k/N)$$

We notice that $n_\lambda^{(N)} = 0$ for $\lambda = i$ and $N = 2, 3,$ and 4, and that $n_\lambda^{(N)} = 1$ for

$\lambda = 1$ and $N = 1, 2, 3;$

$\lambda = i$ and $N = 5, 6, 7, 8;$

$\lambda = -1$ and $N = 2, 3, 4, 5;$

$\lambda = -i$ and $N = 3, 4, 5, 6.$

Consequently the following are identities for the functions $u_k^{(N)}$:

$$u_{4p+3}(j) = 0 \qquad N = 1, 2, 3, 4$$

$$u_{4p}^{(N)}(j)/u_{4p}^{(N)}(0) = (1 - N^{\frac{1}{2}})^{-1} \qquad N = 2, 3$$

$$u_{4p+2}^{(N)}(j)/u_{4p+2}^{(N)}(0) = (1 - N^{\frac{1}{2}})^{-1} \qquad N = 2, 3, 4, 5$$

$$\frac{u_{4p+1}^{(N)}(j)}{u_{4p+1}^{(N)}(1)} = \frac{2 \sin(2\pi j/N)}{2 \sin(2\pi/N) + N^{\frac{1}{2}}} \qquad N = 3, 4, 5, 6$$

and $j \equiv 1 \pmod N$

$$\frac{u_{4p+3}^{(N)}(j)}{u_{4p+3}^{(N)}(1)} = \frac{2 \sin(2\pi j/N)}{2 \sin(2\pi/N) - N^{\frac{1}{2}}} \qquad N = 5, 6, 7, 8$$

and $j \equiv 1 \pmod N$

where p is a positive integer.

We also notice that a linear combination of eigenvalues always exists such that

$$\sum_{k=1}^{M} a_k \underset{\sim}{u}_k^{(N)} = \mathrm{col}\left[1, \frac{1}{1+\lambda N^{\frac{1}{2}}}, \ldots \frac{1}{1+\lambda N^{\frac{1}{2}}}\right] \tag{3.6}$$

where $\lambda = 1$ for $M = n^+$ and $k \equiv 0 \pmod 4$

$\lambda = -1$ for $M = n^-$ and $k \equiv 2 \pmod 4$

and m^+ and n^- are the number of positive and negative eigenvalues respectively.

Since $n_\lambda^{(N)} = 2$ when $\lambda = 1$ for $N = 4, 5, 6, 7$, and when $\lambda = -1$ for $N = 6, 7, 8, 9$, a number α can be found for the above combinations of λ and N and for arbitrary linearly independent $u_k^{(N)}$ and $u_m^{(N)}$, where $k \equiv m \equiv 1 - \lambda \pmod 4$ such that

$$\frac{u_k^{(N)}(j) + \alpha\, u_m^{(N)}(j)}{u_k^{(N)}(0) + \alpha\, u_m^{(N)}(0)} = \frac{1}{1 + \lambda N} \tag{3.7}$$

From (2.8) and (3.3) we have full bases in $E^{(5)}$:

$$E_1^{(5)} = \{e_1 = (1 + 5^{\frac{1}{2}}, 1, 1, 1, 1), e_2 = (0, 1, -1, -1, 1)\}$$

$$E_{-1}^{(5)} = \{e_3 = (1 - 5^{\frac{1}{2}}, 1, 1, 1, 1)\}$$

$$E_{\pm i}^{(5)} = (0, \alpha_\pm, 1, -1, -\alpha_\pm)$$

where $\alpha_\pm = 2^{\frac{1}{2}}(3 - 5^{\frac{1}{2}})^{\frac{1}{2}} \mp 2(5 - 5^{\frac{1}{2}})^{\frac{1}{2}}]$

It is of interest to display a few of these results explicitly:

a. For all integers p and for j = 1, 2, 3, 4:

$$\frac{\sum_{m=-\infty}^{\infty} e^{-\pi(j+5m)^2/5} H_{2+4p}[(j+5m)(2\pi/5)^{\frac{1}{2}}]}{\sum_{m=-\infty}^{\infty} e^{-5\pi m^2} H_{2+4p}[5m(2\pi/5)^{\frac{1}{2}}]} = \frac{1}{1 - \sqrt{5}} \tag{3.8}$$

b. For all integers p and for j = 1 and 2:

$$\frac{\sum_{m=-\infty}^{\infty} e^{-\pi(j+3m)^2/3} H_{2+4p}[(j+3m)(2\pi/3)^{\frac{1}{2}}]}{\sum_{m=-\infty}^{\infty} e^{-3\pi m^2} H_{2+4p}[3m(2\pi/3)^{\frac{1}{2}}]} = \frac{1}{1-\sqrt{3}} \tag{3.9}$$

c. For all integers p and for j = 1 and 2:

$$\frac{\sum_{m=-\infty}^{\infty} e^{-\pi(j+3m)^2/3} H_{4p}[(j+3m)(2\pi/3)^{\frac{1}{2}}]}{\sum_{m=-\infty}^{\infty} e^{-3\pi m^2} H_{4p}[3m(2\pi/3)^{\frac{1}{2}}]} = \frac{1}{1+\sqrt{3}} \tag{3.10}$$

4. Miscellaneous observations prompted by the numerical calculations

We have not investigated the linear dependence of the functions (2.6) beyond noticing that for N=2 through 8 the eigenvectors corresponding to a given eigenvalue are not only linearly dependent, but parallel or antiparallel; that is:

$$u_k^{(N)}(j) = \pm F(N,k)\, u_{k+4p}^{(N)}(j) \tag{4.1}$$

It appears that in certain sets, e.g., $k \equiv 3 \pmod 4$ for

$N = 6$ and 7, and for $k \equiv 1 \pmod 4$ for $N = 4$, eigenvectors with k prime tend to be antiparallel significantly more often than those with k composite, much more than the distribution of primes in the interval $0 \leq k \leq 60$ would suggest.

For primes modulo 8, with the single exception of $k \equiv 3 \pmod 4$, the eigenvectors with $k \equiv 5 \pmod 8$ and $k \equiv 7 \pmod 8$ are antiparallel to the vector for $k \equiv 0$ about four times as frequently as the eigenvectors with $k \equiv 1$ or $3 \pmod 8$. In the exceptional case the 4:1 ratio is reversed.

5. Relationship to theta functions

Writing out the quadratic factor in (2.6), we have:

$$u_k(j) = e^{-\pi j^2/N} \sum_{m=-\infty}^{\infty} e^{-\pi m^2 N - 2\pi m j} H_k[(2\pi)^{\frac{1}{2}}(j + mN)] \tag{5.1}$$

Comparison with the theta function

$$\theta_3(z,t) = \sum_{m=-\infty}^{\infty} e^{-\pi m^2 it + 2miz} \tag{5.2}$$

(Bellman, 1961) shows that the eigenvectors (5.1) are a generalization of these theta functions for the particular arguments $t = iN$ and $z = \pi ij$. As such they may be of mathematical interest.

6. Generating functions for the eigenfunctions

The generating function for the Hermite polynomials is :

$$e^{2xz} - z^2 = \sum_{k=0}^{\infty} H_j(x) \frac{z^k}{k!} \qquad (6.1)$$

(see, for example, Szegö, 1959). Carlitz has pointed out (personal communication, 1970) that it follows from (2.6) and (6.1) that the generating function for the eigenvectors $u_k^{(N)}(j)$ is :

$$\sum_{k=0}^{\infty} u_k(j)\frac{z^k}{k!} = e^{z^2} \sum_{m=-\infty}^{+\infty} \exp\{-\pi[(j+mN)-z\sqrt{2N/\pi}]^2/N\} \qquad (6.2)$$

which can be put in the form :

$$\sum_{k=0}^{\infty} u_k(j)\frac{z^k}{k!} = e^{z^2} \quad u_0(j-z\sqrt{2N/\pi}) \qquad (6.3)$$

7. Expansion of arbitrary vectors in the eigenvectors of W

From the completeness of the functions $e^{-x^2/2} H_k(x)$ in the interval $(-\infty, \infty)$ it follows that the system $\{u_k, k=0, \pm 1,...\}$ is complete. Thus any arbitrary data vector $\underset{\sim}{x}$ can be expanded in terms of these eigenvectors:

$$\underset{\sim}{x} = \sum_{k=1}^{N} a_k \underset{\sim}{u}_k \tag{7.1}$$

where

$$a_k = \underset{\sim}{x} \cdot \underset{\sim}{u}_k \tag{7.2}$$

Now take the finite Fourier transform of both sides of (1.10a):

$$\underset{\sim}{X} = W \underset{\sim}{x} = W \sum_{k=1}^{N} a_k \underset{\sim}{u}_k = \sum_{k=1}^{N} a_k W \underset{\sim}{u}_k = \sum_{k=1}^{N} \lambda_k a_k \underset{\sim}{u}_k \tag{7.3}$$

where $\underset{\sim}{X}$ is the finite Fourier transform of $\underset{\sim}{x}$. Now since $\lambda_k = \pm 1$ and $\pm i$, we see that there is a remarkably simple relation between a periodic function of a discrete variable and its discrete finite Fourier transform: the expansion coefficients of both in terms of the eigenvectors of W have the same absolute value, and $\underset{\sim}{x}$ and $\underset{\sim}{X}$ differ only in the sign and/or the realness of some of the terms in the expansion.

Thus there may exist for discrete time series a canonical form (with respect to the finite Fourier transform) in which the transformation can be done by inspection. It would therefore be of interest to find efficient algorithms for computing the eigenvectors (2.6) and the expansion coefficients

(7.2); it would also be of practical importance to re-examine digital data processing procedures for time series expressed in this canonical form.

A similar statement can be made for the functions (2.3) and the continuous-time Fourier transform (1.1)-(1.2).

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